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OPTIMIZATION OF LINEAR, TIME-INVARIANT SYSTEMS BY MEANS OF QUADRATIC CRITERIA

Thesis for the Degree of Master of Science

by

Hillel Lurie

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LIST OF SYMBOLS

A - n x n invariant system matrix;

B - n x m invariant input matrix;

C - k x m invariant output matrix;

D - n x n invariant closed-loop matrix;

E - 2n x 2n invariant diagonal matrix of (Transl. Note: omission in Hebrew original);

F_1 - n x n invariant matrix (see existence theorem 4.1);

F_2 - n x n invariant matrix (see Eq. (5.5b);

J - n x n invariant matrix, diagonalized according to Jordan;

- 2n x 2n invariant matrix (see Eq. (3.22));

I - set of eigenvectors;

M - 2n x 2n invariant matrix of the Euler-Lagrange equations;

P - n x n matrix of the solution of the Riccati equation;

Q - n x n definite or semidefinite matrix (see Eq. (3.11));

S - n x n matrix defined by Eq. (5.23);

R_1 - k x k definite (positive or negative, depending on the case) matrix;

R_2 - m x m positive definite matrix;

T - n x n invariant matrix defined by Eq. (3.4);

V - 2n x 2n invariant matrix;

X - n x n invariant matrices;

Y - n x n invariant matrices;

Z - n x n invariant matrices;

— e_{ij} (jw) - eigenvector corresponding to eigenvalue P = jw;

e - n-dimensional vector, eigenvector of M;

f - quadratic cost function;

p - eigenvalue of M

$i, j, k, l, m, n, q, r, s$ - current subscripts;

$v(jw)$ - eigenvector of M' corresponding to eigenvalue $p = jw$;

t - current time;

t_f - optimum-process time;

t_c - conjugate-point time;

u - m-dimensional vector - input vector;

y - k-dimensional vector - output vector;

x - n-dimensional vector - vector of state variables;

λ - n-dimensional vector - vector of Lagrangian multipliers;

$\Theta(t)$ - $2n \times 2n$ time-variant transformation matrix;

jw - pure imaginary eigenvalue of M

α - n-dimensional vectors;

β - n-dimensional vectors;

μ - n-dimensional vectors;

η - n-dimensional vectors;

$\mathcal{C}^n$ - n-dimensional space;

$\mathcal{R}(U)$ - transformation range corresponding to the matrix subspace;

$\mathcal{R}^{\perp}(U)$ - subspace orthogonal to $\mathcal{R}(U)$;

$\mathcal{N}(U)$ - nullity subspace of U;

U^+ - pseudo inverse of U;

U_{ob} - square submatrix of U whose columns and rows correspond to observable state variables of A;

U_{nob} - square submatrix of U whose columns and rows correspond to nonobservable state variables of A;

U_c - square submatrix of U whose columns and rows correspond to controllable state variables of A;

U_{nc} - square submatrix of U whose columns and rows correspond to noncontrollable state variables of A.

ABSTRACT

This thesis is concerned with optimization of linear, time-invariant systems by means of quadratic criteria.

Given a system described by its state variables:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(o) &= x_o \\ y &= Cx \end{aligned} \tag{1}$$

it is required to find an optimal controller u(t) that will minimize the cost function

$$f(u;x_o) = \int_{t_o}^{t_f} (y'(t)R_1 y(t) + u'(t)R_2(t)u(t)) \, dt \tag{2}$$

where R_1 and R_2 are symmetric matrices.

It is the purpose of this thesis to find necessary and sufficient conditions for the existence of an optimal solution and to find the properties of such a solution if it does exist. The method of solution is based on the calculus of variations and analysis of the results thus obtained by the methods of linear algebra.

Chapter 2 surveys the general results obtained by others concerning the existence of a solution to this problem [1, 2]. It is found that necessary and sufficient conditions for the existence of an optimal solution are the following:

a. $R_2 > 0$;

b. Satisfaction of the Euler-Lagrange equations

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & BR_2^{-1}B' \\ -C'R_1C & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = M \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{3}$$

with the boundary conditions

$$\lambda(t_f) = 0 ; \quad x(t_o) = x_o$$

c. Nonexistence of a point t_c, which would be conjugate to t_f over the range (t_0, t_f).

Chapters 3 through 5 are concerned with the application of these conditions to the case of $R_1 > 0$ and with the determination of the constraints on the quantities A, B, and C and on their interrelationships. Chapter 3 discusses the special properties of a matrix M. It is found that the eigenvalues of this matrix are symmetric even with respect to the imaginary axis. The fact is also established that, with the exception of special cases, the matrix M has no eigenvalues p such that $Re(p) = 0$.

It turns out that, with the exception of these particular cases, it is possible to find a matrix E such that

a.
$$E^{-1}ME = \begin{bmatrix} J & 0 \\ 0 & -J' \end{bmatrix} \tag{4}$$

where J is a matrix in Jordan canonical form

b.
$$E^{-1} = \begin{bmatrix} E'_{22} & -E'_{12} \\ -E'_{21} & E'_{11} \end{bmatrix} \tag{5}$$

It should be noted that the thesis also discusses the exceptional cases when eigenvalues of the type $Re(p) = 0$ do exist.

Chapter 4 analyzes the properties of matrices E_{ij}. Necessary and sufficient conditions are found for the existence of matrices E_{ij}^{-1}. When these do not exist, $\mathfrak{N}(E_{ij})$; and $\mathfrak{R}(E_{ij})$ are found.

Chapter 5 utilizes the information obtained in Chapters 3 and 4 for determination of the nature of the optimal solution and for establishing the existence or nonexistence of a conjugate point. This chapter proves the known results that, in the case of $R_1 > 0$: a) an optimal solution exists for every system for finite t_f; b) a solution exists as $t_f \to \infty$ only if the observable part of A can be stabilized.

The method of solution presented here is not new and has already been used in [1, 2]. However, this approach was used there on the assumption that a solution does exist [without proving this existence]. Because of this, some assumptions were made without proof and without establishing limits of applicability of these assumptions. In addition, the above studies were restricted to cases when M is a simple linear transformation and has no eigenvalues p such that $Re(p) = 0$. The present thesis is concerned also with cases excluded by the above restrictions.

The method presented here has an advantage over the classical approach of [3] in that it is not restricted to a specific property of the problem, but addresses itself directly to the defining Euler-Lagrange equations. This is illustrated in Chapter 6, where the problem is solved for the case $R_1 < 0$, which is termed the maximization problem and cannot be proven by the method of [3]. In this Chapter, we refer to Chapters 3-5 in order to elucidate the special properties of M in this case and to point out the differences between the nature of the minimization and maximization solution.

The results in this cases are more quantitative than qualitative. Precise necessary and sufficient conditions were obtained only for the limiting case as $t_f \to \infty$; these are:

a. $$x^{*'}\left\{ (A-j\omega I)\,(C'R_1C)^{\dagger}\,(A'+j\omega I) + B'R_2B \right\} x < 0$$
$$\forall \; x : (A'+j\omega I)\, x \in \mathcal{R}(C'R_1C)$$
$$\forall \; \omega$$

b. The observable part of A is unconditionally stable.

Only partial results are obtained for finite t_f. These results are quantitative and are given in the thesis proper.

CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem of the Controller

One of the most frequent problems in the theory of optimal control is that of the controller. This problem pertains to the following. We are given a system with initial conditions at time t_0. It is required to select a controller or an input signal which will bring the system to the vicinity of its zero state upon reaching a specified time t_f. The problem is solved by establishing an appropriate cost function for the state of the system at any given instant and by finding an input signal which will result in minimization of the cost function.

When no *apriori* restrictions are imposed on the input signal, it is customary to supplement the cost function by a term the minimization of which will cause the input signal to attain a sensible amplitude or energy. In fact, this problem is a particular case of the optimization problem as a whole. Since this problem has a practical bearing on many applications, it has been treated extensively in the literature of control theory.

1.2 Statement of the Maximization Problem

Another problem of this kind is that of maximization. Here we are given a system with initial conditions specified at t_0. It is necessary to select an input signal which will result in maximization of the state variables or some of them up to a given time t_f. Again in this case the cost function is supplemented by a term which limits the input signal to a finite value.

The requirements of maximization of the state variables on the one hand and of minimization of the input signal on the other are not always compatible. For this reason, the existence of a final solution to this problem is not *a priori* guaranteed. Problems of this kind have a practical bearing on the design of systems using the method of design for the worst case under the assumption of maximum possible interference. Similar maximization problems are also encountered in problems of evasion.

1.3 Method of Solution

This study is limited to time-constant linear systems and to time-constant quadratic cost functions. These conditions make it possible to obtain relatively simple conditions for the existence of a solution and for analysis of the significance of these conditions. The method of solution is based on the calculus of variations and on examination of the Euler-Lagrange equations which are obtained in it by using linear-algebra techniques.

1.4 Discussion Published Studies

The above method of solution is not new and has already been examined in a number of articles [1, 2]; the principal contribution of this paper is the proof of assumptions made in the above studies, but not proven there. In addition, this study pertains to cases not discussed elsewhere. The present study is based (in part) on the classical conditions for the existence of a solution, which are presented here without proof and which were taken from [4, 5].

The bulk of the mathematical computations uses the methods of linear algebra. In conjunction with this, extensive use is made of [8]. The variational computations are based on a number of basic references.

1.5 Scope of the Study

As was mentioned, the principal contribution of this paper is proof of the assumptions used in [1, 2, 3] without any proof. This proof is given in Chapters 4 and 5. Chapter 3 serves as an introduction to these two chapters. Chapter 6 considers the problem of the maximum and it points to the differences between the solutions of the maximization and minimization problems.

CHAPTER 2.

STATEMENT OF THE PROBLEM AND ITS SOLUTION

2.1 Statement of Problem

We have given a time-invariant linear system of finite order n defined by the linear vectorial equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(t_o) &= x_o \\ y &= Cx \end{aligned} \tag{2.1}$$

where x(t) is an n-dimensional vector describing the state of the system at time t, u(t) is an m-dimensional vector serving as the input signal at time t, and y(t) is a k-dimensional vector serving as the output signal at time t.

Every piecewise-continuous input signal will be called admissible. A, B, and C are k x n, n x m and n x n matrices respectively.

Together with the above system, we are given the quadratic cost function

$$f(u;x_o) = \int_{t_o}^{t_f} (y'(t)R_1y(t) + u'(t)R_2u(t)) \quad dt \tag{2.2}$$

where t_0 and t_f are specified.

R_1 and R_2 are k x k and m x m matrices respectively. It is required to find an optimal input signal $u^*(t)$ which will minimize the cost function (2.2) subject to condition (2.1), i.e., $f(u^*) \leq f(u)$ for all u(t).

2.2 Presentation of Solution

The solution of the problem and conditions for existence of a solution are summarized in the theorems that follow. The proof of these is given in [1, 4].

Theorem 1 The optimum input signal u(t) for system (2.1) and cost function (2.2) satisfies the Euler-Lagrange equations:

$$\dot{x}(t) = Ax(t) + B\,u(t) \tag{2.3}$$

$$\dot{\lambda}(t) = -A'\lambda(t) - C'R_1Cx(t) \tag{2.4}$$

$$R_2\,u(t) = -B'\lambda(t) \tag{2.5}$$

with the boundary conditions

$$x(t_o) = x_o \tag{2.6}$$

$$\lambda(t_f) = 0 \tag{2.7}$$

where the n-dimensional vector $\lambda(t)$ is the vector of the Lagrangian multipliers. This theorem shall be termed necessary condition I for the existence of a solution.

Theorem 2 (Necessary condition II). A necessary condition for the existence of a unique solution of this problem is that $R_2 > 0$. This condition is analogous to the Legendre-Clebsch condition in the calculus of variations.

Theorem 3 (Necessary condition III). A necessary condition for the existence of a solution to this problem is the nonexistence of a point t_c conjugate to the point t_f in the interval (t_{00}, t_f). This condition is analogous to the Jacobi condition in the calculus of variations.

Definition of conjugate point. A point t_c is conjugate to t_f if it is contained in the interval (t_0, t_f) and there exists a nontrivial solution to Eqs. (2.3)-(2.5) which satisfies the boundary conditions

$$x(t_c) = 0 \tag{2.8}$$

$$\lambda(t_f) = 0 \tag{2.9}$$

Theorem 4 Satisfaction of the necessary conditions I, II, and III is a sufficient condition for the existence of a solution to the problem for finite t_f.

2.3 Solution of Euler-Lagrange Equations

It is possible to write Eqs. (2.3)-(2.5) as a system of linear homogeneous equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR_2^{-1}B' \\ -C'R_1C & -A' \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{2.10}$$

If we define the matrix

$$M = \begin{bmatrix} A & -BR_2^{-1}B' \\ -C'R_1C & -A' \end{bmatrix} \tag{2.11}$$

equation (2.10) may be written

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{2.12}$$

The transformation matrix of a homogeneous equation will be denoted by $\theta(t_1, t_2)$:

$$\Theta(t_1;t_2) = \begin{bmatrix} \Theta_{11}(t_1;t_2) & \Theta_{12}(t_1;t_2) \\ \Theta_{21}(t_1;t_2) & \Theta_{22}(t_1;t_2) \end{bmatrix} \tag{2.13a}$$

Since M is a time-invariant matrix, we have

$$\Theta(t_1;t_2) = \Theta(t_1-t_2) \tag{2.13b}$$

The submatrices θ_{ij} are n x n, but have no properties of the transformation matrix. Using Eqs. (2.13) and (2.6), we get

$$\begin{aligned} x(t) &= \Theta_{11}(t-t_o)x_o + \Theta_{12}(t-t_o)\lambda_o \\ \lambda(t) &= \Theta_{21}(t-t_o)x_o + \Theta_{22}(t-t_o)\lambda_o \end{aligned} \tag{2.14}$$

and, from boundary conditions (2.7), we get

$$\Theta_{21}(t_f-t_o)x_o + \Theta_{22}(t_f-t_o)\lambda_o = 0 \tag{2.15}$$

2.4 Check for Nonexistence of a Conjugate Point

Substitution of boundary conditions (2.7) and (2.8) into Eqs. (2.13) yields

$$\Theta_{22}(t_f-t_c)\ \lambda(t_c) = \lambda(t_f) = 0 \tag{2.16}$$

For this equation to have only a trivial solution, it is necessary that

$$\det \{ \Theta_{22}(t_f - t_0) \} \neq 0 \tag{2.17}$$

This condition is analogous to the condition of existence of a unique solution of Eq. (2.15).

2.5 Solution in a Closed Loop

When there is no conjugate point, it is possible to solve the problem in a closed loop, with $\lambda(t)$ selected at any given time in such a manner that the value of $x(t)$ at that time is the initial condition of the system and it is necessary to minimize the cost function in the interval (t_0, t_f). From Eq. (2.15)

$$\lambda(t) = \Theta_{22}^{-1}(t_f - t) \quad \Theta_{21}(t_f - t)x(t) \tag{2.18}$$

$$\lambda(t) = P(t) \; x(t)$$

where

$$P(t) = \Theta_{22}^{-1}(t_f - t) \quad \Theta_{21}(t_f - t) \tag{2.19}$$

The matrix P(t) satisfies the Riccati equation

$$\dot{P}(t) = -C'R_1C - A'P(t) - P(t)A + P(t)BR_2B'P(t); \; P(t_f) = 0 \tag{2.20}$$

and the value of the cost function is

$$f(x_0; t_f) = x_0' \, P(t_f - t_0)x_0 \tag{2.21}$$

A solution will exist as $t_f \to \infty$ if there exists a solution for every finite t_0 and if

$$\lim P(t_f - t_0) = \text{finite} \;, \; (t_f - t_0) \to \infty$$

CHAPTER 3

ANALYSIS OF THE EULER-LAGRANGE EQUATIONS

3.1 Introduction

The following chapters are concerned with solution of the problem of the minimum, i.e., in the case of $R_1 > 0$. As was shown in the preceding chapter, the optimum solution satisfies the Euler-Lagrange equations with appropriate boundary conditions.

In the case of time-invariant linear systems and quadratic cost functions with time-constant coefficients, these equations are defined by the system of time-invariant linear equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{2.12}$$

$$M = \begin{bmatrix} A & -BR_2^{-1}B' \\ -C'R_1C & -A' \end{bmatrix} \tag{2.11}$$

We shall attempt to gather a maximum of information on the nature of the optimal solution by analyzing the above equations by the methods of linear algebra.

In the proofs which follow, we shall on a number of occasions need the canonical forms of A, B, and C. For each of the systems A, B, and C, it is possible, after appropriate transformation, to obtain the following form (see [8], Chapter 11):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \tag{3.1}$$

$$y = \begin{bmatrix} 0; & C_2; & 0; & C_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where the matrix elements A pertaining to observable, nonobservable, controllable, and noncontrollable state variables are as follows:

$$A_{ob} = \begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}; \quad A_{nob} = \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix} \tag{3.2}$$

$$A_c = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}; \quad A_{nc} = \begin{bmatrix} A_{33} & A_{34} \\ 0 & A_{44} \end{bmatrix}$$

Henceforth, it shall be assumed that the matrices are initially given in this canonical form. Moreover, we assume that A_{ij} have already been given in Jordan canonical form.

3.2 The Eigenvalues of the Matrix M

Theorem 5 The Matrix M has the Property

$$M' = T^{-1}MT \tag{3.3}$$

where

$$T = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{3.4}$$

The proof of the above is obtained immediately by substituting the values of the matrices M and T. The matrix T is a special matrix with the property

$$T' = -T = T^{-1} \tag{3.5}$$

This property will be found useful in what follows.

Theorem 6 If p_i is an eigenvalue of multiplicity r, then $-p_i$ is also an eigenvalue of M of multiplicity r. If $p_i = 0$, then it has an even multiplicity

Proof: the eigenvalues are the solutions of

$$
\begin{aligned}
\det(M-p_i I) &= 0 \\
\det(M'-p_i I) &= 0
\end{aligned}
\tag{3.6}
$$

In this case

$$
\begin{aligned}
\det T(M'-p_i I)T^{-1} &= 0 \\
\det(M+p_i I) &= 0
\end{aligned}
\tag{3.7}
$$

i.e., if p_i is an eigenvalue of M of multiplicity r, then $-p_i$ is also an eigenvalue of M of multiplicity r.

Since M is an 2n x 2n matrix, it will have 2n eigenvalues (not necessarily distinct). Each eigenvalue has corresponding to it another eigenvalue of an opposite sign (but same magnitude). Therefore if it contains a zero eigenvalue, it will have another zero eigenvalue corresponding to it, so that the total number of eigenvalues will be even (2n).

Theorem 7 If p_i is an eigenvalue of M of multiplicity r and it has corresponding to it q_i eigenvectors corresponding to each of which has [in turn] are $s_{iq} \geq 0$ generalized eigenvectors, then also the $-p_i$ of multiplicity r will have corresponding to it q_i eigenvectors, and corresponding to each such vector are s_{iq} generalized eigenvectors.

Proof: Using Jordan canonization, we can see that the multiplicities of the eigenvectors, the number of these eigenvectors, and the number of the generalized eigenvectors of M corresponding to them are equal to those of M. Therefore,

$$
\begin{aligned}
(M-p_i I)^{s_{iq}} \quad & e_{ik} = 0, \\
T^{-1}(M'-p_i I)^{s_{iq}} T T^{-1} \quad & e_{ik} = 0 \\
(T^{-1}M'T-p_i I)^{s_{iq}} T^{-1} \quad & e_{ik} = 0 \\
-(M+p_i I)^{s_{iq}} T^{-1} \quad & e_{ik} = 0
\end{aligned}
$$

If we write $Te_{ik}^{-1} = e_{-ik}$, we can see that, for each eigenvector (ordinary or generalized) of M corresponding to p_i, there exists an eigenvector (ordinary or generalized) of M corresponding to $-p_i$.

This theorem points to symmetrical properties in the spectral analysis of M. This symmetry has no meaning for $p_i = 0$. It will be seen further on that this case requires special consideration.

3.3 Check for the Existence of Eigenvalues with Re(p) = 0

Let us now check whether there exist eigenvalues with zero real part. It appears that because $R_1 > 0$ eigenvalues like this may exist in special cases.

Theorem 8 The matrix M will have eigenvalues with zero real part if and only if the noncontrollable part of A in A and B or the nonobservable part of A in A and C contain eigenvalues with zero real part.

Proof: a) We write out in detail the matrix M, with A defined as in Eq. (3.1):

$$M = \left[\begin{array}{cccc|cccc} A_{11} & A_{12} & A_{13} & A_{14} & B_1R_2^{-1}B_1' & B_1R_2^{-1}B_2' & 0 & 0 \\ 0 & A_{12} & 0 & A_{24} & B_2R_2^{-1}B_1' & B_2R_2^{-1}B_2' & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -A_{11}' & 0 & 0 & 0 \\ 0 & C_2'R_1C_2 & 0 & C_2R_1C_4 & -A_{12}' & -A_{22}' & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{13}' & 0 & -A_{33}' & 0 \\ 0 & C_4'R_1C_2 & 0 & C_4'R_1C_4 & -A_{14}' & -A_{24}' & -A_{34}' & -A_{44}' \end{array}\right] \quad (3.8)$$

Changing the order of the state variable of this matrix, we arrive at the matrix M^*, which has eigenvalues identical to those of the matrix M and which has the form

$$M^* = \left[\begin{array}{cccccccc} A_{11} & A_{13} & 0 & A_{12} & A_{14} & B_1R_2^{-1}B_1' & B_1R_2^{-2}B_2' & 0 \\ 0 & A_{33} & 0 & 0 & A_{34} & 0 & 0 & 0 \\ 0 & 0 & -A_{44}' & C_4'R_1C_2 & C_4'R_1C_4 & -A_{14}' & -A_{24}' & -A_{34}' \\ 0 & 0 & 0 & A_{22} & A_{24} & B_2R_2^{-1}B_1' & B_2R_2^{-1}B_2' & 0 \\ 0 & 0 & 0 & 0 & A_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_{11}' & 0 & 0 \\ 0 & 0 & 0 & C_2'R_1C_2 & C_2'R_1C_4 & -A_{12}' & -A_{22}' & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_{13}' & 0 & -A_{33}' \end{array}\right] \quad (3.9)$$

It is clear from this matrix that the eigenvalues of A_{11}, A_{33}, and A'_{44} are also the eigenvalues of M. According to Theorems 6 and 7, the eigenvalues of A_{44} will also be the eigenvalues of M; hence if A_{11}, A_{33}, or A_{44} are eigenvalues with zero real part, M will also have such eigenvalues. From this, the eigenvalues with zero real part (and also others) of the noncontrollable and/or nonobservable part of A are also the eigenvalues of M.

b) Let us assume that M has an eigenvalue p = jw. Its corresponding eigenvector $\begin{bmatrix} x \\ \lambda \end{bmatrix}$ will satisfy the conditions

$$\begin{aligned} (A - j\omega I)\, x - BR_2^{-1}B'\,\lambda &= 0 \\ -C'R_1C\, x - (A' + j\omega I)\lambda &= 0 \end{aligned} \tag{3.10}$$

The matrix $Q = C'R_1C$ is symmetrical. Accordingly, we divide space $\mathcal{E}^n$ into two complementary subspaces

$$\mathcal{C}^n = \mathcal{R}(Q) \overset{\perp}{\oplus} \mathcal{N}(Q) \tag{3.11}$$

The vector $x \in \mathcal{C}^n$ can be decomposed into two components

$$x = x_{ob} + x_{nob} \begin{cases} x_{nob} = \in \mathcal{N}(Q) \\ x_{ob} = \in \mathcal{R}(Q) \end{cases} \tag{3.12}$$

Analogously, we decompose $(A' + jwI)\lambda$:

$$\begin{aligned} \lambda_{ob} &: (A' + j\omega I)\lambda_{ob} \in \mathcal{R}(Q) \\ \lambda_{nob} &: (A' + j\omega I)\lambda_{nob} \in \mathcal{N}(Q) \end{aligned} \tag{3.13}$$

It is clear from Eq. (3.6) that

$$(A' + j\omega I)\ \lambda_{nob} = 0 \tag{3.14}$$

For the system composed of A and C, in which the nonobservable part of A contains eigenvalues of the kind p = jw, λ_{nob} will have a nonzero solution. For all the other cases

$$\lambda_{nob} = 0$$

$$\lambda = \lambda_{ob}$$

and from the second of equations (3.10)

$$x_{ob} = -Q^{\dagger} (A+j\omega I) \lambda_{ob} \tag{3.15}$$

We substitute x_{ob} into Eq. (3.6) and multiply it from the left by λ^*_{ob}, thus obtaining

$$-\lambda^*_{ob}\left\{(A-j\omega I)Q^{\dagger}(A'+j\omega I)-BR_2^{-1}B'\right\}\lambda_{ob} = 0 \tag{3.16}$$

this, because

$$\lambda^*(A-j\omega I)x_{nob} = \left\{x^*_{nob}(A'+j\omega I)\lambda\right\}^{*} = (x'^*_{nob}Qx_{ob})^* = 0 \tag{3.17}$$

Expression (3.12) contains two expressions which are at least semidefinite. Writing A, B, and C in their canonical form, we see that Eq. (3.12) will be semidefinite only when the observable and noncontrollable parts of A in A, B, and C contain eigenvalues of the kind Re(p) = 0. In this case, there exists a $\lambda \neq 0$ which satisfies Eq. (3.12). In all other cases Eq. (3.12) is fully definite, and hence the only solution for λ_{ob} is $\lambda_{ob} = 0$.

Subdivision of $\mathcal{C}^n$ into the two complementary subspaces

$$\mathcal{C}^n = \mathcal{R}(BR_2^{-1}B') \overset{\perp}{\oplus} \mathcal{N}(BR_2^{-1}B') \tag{3.18}$$

and repetition of operations similar to (3.11)-(3.17) makes it possible to prove that similar results are obtained for the lower part of the vector x: $x \neq 0$ with respect to vectors corresponding to eigenvalues which are also eigenvalues of the noncontrollable or nonobservable part of A; $x = 0$ for all the other cases.

It follows from this that if jw is not a noncontrollable or nonobservable eigenvector of A, then the eigenvector corresponding to it is identically zero, i.e., jw is not an eigenvalue of M.

Conclusion: The only eigenvalues of M of the kind jw are those belonging to noncontrollable or nonobservable parts of A (provided that such exist).

Remark: It can be seen from the canonical forms of A, B, and C that the following holds when M has jw as an eigenvalue: if the eigenvalue belongs to the nonobservable part, it will have type-a eigenvectors: $x \neq 0$ the eigenvector of A and $\lambda = 0$. Type-b eigenvectors $x \neq 0$, $\lambda \neq 0$, whose components correspond to nonobservable state variables, comprise the eigenvector of A'_{nob}. If it belongs to the noncontrollable part of A, then there will exist type-c eigenvectors; $x = 0$ and $x \neq 0$ which are the eigenvector of A'. Type-d eigenvectors: $x \neq 0$, whose components corresponding to noncontrollable state variables, comprise the eigenvector of A_{nc}, and $\lambda \neq 0$.

Note that while type a and b vectors are generally simple eigenvectors, type -c and -d vectors are generalized eigenvectors. This is now shown by a number of numerical examples.

Example 3.1. We have the system defined by matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$R_1 = R_2 = I$$

where A has nonobservable eigenvalues $p = \pm jw$. The matrix M then has the form

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 3 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

This matrix has eigenvalues $\pm\sqrt{2}$ and $\pm j$ of multiplicity two. The eigenvectors will be: $e(+\sqrt{2}), e(-\sqrt{2})$ and also:

$$e_1(+j) = \begin{bmatrix} 0 \\ 1 \\ -j \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_1(-j) = \begin{bmatrix} 0 \\ 1 \\ +j \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_2(+j) = \begin{bmatrix} 2/3\ j \\ 0 \\ 5/3 \\ 2/3(1+j) \\ -2/3 \\ -2/3\ j \end{bmatrix} \quad e_2(-j) = \begin{bmatrix} -2/3\ j \\ 0 \\ 5/3 \\ 2/3(1-j) \\ -2/3 \\ 2/3\ j \end{bmatrix}$$

Vectors $e_1(\pm j)$ are type-a vectors, while vectors $e_2(\pm j)$ are type-b vectors, which satisfy

$$(M \mp j)\, e_2\,(\pm j) = -e_1\,(\pm j)$$

Example 3.2. We have the system defined by the matrices:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$R_1 = R_2 = I$$

A has a noncontrollable eigenvalue $p = \pm jw$. The matrix has the form

$$M_2 = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -3 & +1 & 0 \end{bmatrix}$$

This matrix, as well as matrix M_1 in Example 3.1, has eigenvalues $\pm\sqrt{2}$ and $\pm j$ of multiplicity two. The eigenvectors are $e(+\sqrt{2})$, $e(-\sqrt{2})$. Also,

$$e_1(+j) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -j \end{bmatrix} \quad e_1(-j) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ +j \end{bmatrix} \quad e_2(+j) = \begin{bmatrix} -2/3(1-j) \\ 2/3 \\ 2/3\, j \\ +2/3\, j \\ 0 \\ 1/3 \end{bmatrix} \quad e_2(-j) = \begin{bmatrix} -2/3(1+j) \\ 2/3 \\ -2/9\, j \\ -2/3\, j \\ 0 \\ 1/3 \end{bmatrix}$$

In this case $e_1(\pm j)$ are type-c vectors, while $e_2(\pm j)$ are type-d vectors which satisfy

$$e_1(\pm j)\; e_2(\pm j) = -\,e_1(\pm j)$$

3.4 Canonization of M

Let us now apply to M a transformation which will bring it to a canonical form similar to the Jordan canoninical form. In order to obtain the required transformation, we shall subdivide the eigenvectors (ordinary or generalized) of M into two groups as follows:

1. Group I(+). This group contains all the eigenvectros (ordinary and generalized) corresponding to the eigenvalues which satisfy the condition Re(p) > 0. The vectors will be selected so as to satisfy the equations

$$
\begin{aligned}
Me_{iq1} &= p_i e_{iq1} \\
Me_{iq2} &= p_i e_{iq2} + e_{iq1} \\
&- - - - - - - - - - \\
Me_{iqs_{iq}} &= p_i e_{iqs_{iq}} + e_{iq(s_{iq}-1)}
\end{aligned}
\tag{3.19}
$$

2. Group I(-). This group contains all the eigenvectors corresponding to the eigenvalue satisfying the condition Re(p) < 0. The vectors will be selected so as to satisfy the equations:

$$
\begin{aligned}
Me_{jq1} &= p_j e_{jq1} - e_{jq2} \\
&- - - - - - - - - - \\
Me_{jq(s_{jq}-1)} &= p_j e_{jq(s_{jq}-1)} - e_{jqs_{jq}} \\
Me_{jqs_{jq}} &= p_j e_{jqs_{jq}}
\end{aligned}
\tag{3.20}
$$

3. Ordinary or generalized eigenvectors corresponding to eigenvalues of the kind Re(p) = 0 exist, as was mentioned above, only when A has eigenvalues such that the eigenvectors corresponding to them belong to the noncontrollable or nonobservable subspace.

a. When the eigenvalue satisfying the condition Re(p) = 0 belong to the nonobservable part of A, we shall refer vectors in the form $\begin{bmatrix} x \\ o \end{bmatrix}$ to group I(+) and the others, corresponding to the same eigenvalue, will be refered to group I(-).

b. When the eigenvalue belongs to the noncontrollable part of A, we shall refer vectors of the form $\begin{bmatrix} o \\ x \end{bmatrix}$ to group I(-), while the remaining vectors, corresponding to the same eigenvalue, will be referred to group I(+).

Having established the groups I(+) and I(-), we now construct the matrix:

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \tag{3.21}$$

so that

$$\text{Columns of } \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} \in I_{(+)}$$

$$\text{Columns of } \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \in I_{(-)}$$

It can be seen from Eqs. (3.22) and (3.24) that, with the exception of the case of $p_i = jw$, all the generalized eigenvectors interconnected by a given chain are contained in this group. We therefore can, by proper arrangement of the columns of the matrix E, find a matrix which will satisfy

$$E^{-1}ME = \mathcal{J} \tag{3.22}$$

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} = \begin{bmatrix} +J & -J_{nob}(j\omega) \\ -J_{nc}(j\omega) & -J' \end{bmatrix} \tag{3.23}$$

J being a Jordan matrix.

The use of $J_{nob}(jw)$ and $J_{nc}(jw)$ is mandatory since, if there exist eigenvalues of the type $Re(p) = 0$, then the generalized eigenvectors related to them will belong to both groups.

We now define the matrix $J_{nob}(jw)$. The matrix $J_{nc}(jw)$ is defined similarly by replacing observability terminology with controllability terminology. The general form of J_{nob} will be

$$J_{nob}(j\ ) = \left[\begin{array}{c|c|ccc} 0 & & 0 & 0 & 0 \\ \hline \hline 0 & & 0 & 1 & 0 \\ 0 & & 1 & 0 & 0 \\ 0 & & 0 & 0 & 1 \end{array}\right] \tag{3.24}$$

The components of $J_{nob}(jw)$ will be determined according to the following rules:

$$
\begin{array}{llll}
 & & & J_{nob}(j\omega) = J'_{nob}(j\omega) \\
 & Re(p_j) = 0 \ \text{או} & Re(p_i) = 0 & J_{nob_{ij}}(j\omega) = 0 \\
p_k = p^*_l \ ; & Im(p_k) = 0 \ ; & Re(p_k) = 0 \ ; & J_{nob_{kl}}(j\omega) = 1 \\
 & p_n = p^*_m & ; & J_{nob_{mn}}(j\omega) = 0 \\
 & p_r = 0 & ; & J_{nob_{rr}}(j\omega) = 1 \\
 & p_q = 0 & ; & J_{nob_{qq}}(j\omega) = 0
\end{array}
$$

and this when i, j, k, l, m, n, r, and q correspond to nonobservable state variables of A.

We shall now illustrate the construction of groups I(+) and I(-) and the structure of the matrix J by means of several examples.

Example 3.3 The eigenvectors in Example 3.1 corresponding to the eigenvalues j and -j are subdivided according to this classification into groups I(+) and I(-) as follows:

I (+)	to group	$e_1(-j)$;	$e_1(+j)$;	$e(+\sqrt{2})$
I (-)	to group	$e_2(-j)$;	$e_2(+j)$;	$e(-\sqrt{2})$

Example 3.4 The eigenvectors in Example 3.2 corresponding to eigenvalues j and -j are subdivided into groups as follows:

I (-)	to group	$e_1(-j)$;	$e_1(+j)$;	$e(-\sqrt{2})$
I (+)	to group	$e_2(-j)$;	$e_2(+j)$;	$e(+\sqrt{2})$

Example 3.5 The matrix $\mathcal{J}$ corresponding to Example 3.1 is

$$
\left[\begin{array}{ccc|ccc}
+\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -j & 0 & 0 & 0 & 1 \\
0 & 0 & +j & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & +j & 0 \\
0 & 0 & 0 & 0 & 0 & -j
\end{array}\right]
$$

Example 3.6 The matrix $\mathcal{Y}$ corresponding to Example 3.2 is

$$
\left[\begin{array}{ccc|ccc}
+\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -j & 0 & 0 & 0 & 0 \\
0 & 0 & +j & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 1 & 0 & +j & 0 \\
0 & 1 & 0 & 0 & 0 & -j
\end{array}\right]
$$

Theorem 5. It is possible to find from matrices $E^{-1}ME = \mathcal{Y}$ one which satisfies

$$E = T^{-1}E'^{-1}T \tag{3.25}$$

Proof:

$$E^{-1}ME = \mathcal{Y}$$

$$T^{-1}E^{-1}TT^{-1}MTT^{-1}ET = T^{-1}\mathcal{Y}\,T$$

and by transposition

$$(T^{-1}E'^{-1}T)^{-1}M(T^{-1}E'^{-1}T) = \mathcal{Y}$$

i.e., the matrix $T^{-1}E'^{-1}\overline{T}$ also satisfies Eq. (3.22)

We write

$$E^{-1\,\prime} = V'$$

$$T^{-1}E'^{-1}T = \begin{bmatrix} V'_{22} & -V'_{12} \\ -V'_{21} & V'_{11} \end{bmatrix}$$

and construct the matrix

$$\overline{E} = \begin{bmatrix} E_{11} & -V'_{12} \\ E_{21} & V'_{11} \end{bmatrix}$$

It is clear that $\bar{E}$ also is capable of putting M into canonical form according to Eq. (3.22). For this matrix we have

$$T^{-1}\bar{E}'T\bar{E} = \begin{bmatrix} V_{11} & V_{12} \\ -E'_{21} & E'_{11} \end{bmatrix} \begin{bmatrix} E_{11} & -V'_{12} \\ E_{21} & V'_{11} \end{bmatrix} = I \tag{3.26}$$

This because each column in $\begin{bmatrix} -V'_{12} \\ V'_{11} \end{bmatrix}$ is a vector contained in the subspace perpendicular to the subspace defined by columns $\begin{bmatrix} V'_{11} \\ V'_{12} \end{bmatrix}$, i.e.,

$$\bar{E}^{-1} = \begin{bmatrix} \bar{E}'_{22} & -\bar{E}'_{12} \\ -\bar{E}'_{21} & \bar{E}'_{11} \end{bmatrix} \tag{3.27}$$

Subsequently we shall refer only to the matrix $\bar{E}$, dropping the bar for brevity and calling it simply E.

From Eq. (3.25) we get the following system of equations

$$\begin{aligned} E'_{22}E_{12}-E'_{12}E_{22} &= 0 & \qquad E_{21}E'_{22}-E_{22}E'_{21} &= 0 \\ E'_{21}E_{11}-E'_{11}E_{21} &= 0 & \qquad E_{11}E'_{22}-E_{12}E'_{21} &= I \\ E_{11}E'_{12}-E_{12}E'_{11} &= 0 & \qquad E'_{22}E_{11}-E'_{12}E_{21} &= I \end{aligned} \tag{3.28}$$

$$E'_{12}E_{21}-E_{21}E'_{12} = 0$$

CHAPTER 4

EXISTENCE THEOREMS

This chapter examines the properties of the eigenvector matrix E and those of its submatrices. These properties are of great importance in checking for the existence of a solution. Not all the theorems given here are needed for solving the problem but are presented here to give a more complete picture of the properties of eigenvectors.

The existence theorems which will be proven here depend directly on the apparently arbitrary structure of the matrix E as it was defined in Eqs. (3.22) and (3.27). This structure was selected in order to impart to these theorems more of a physical meaning. It is clear that, if we had selected the vectors of E in a different manner, we would have obtained different theorems.

Existence theorem 4.1

a. If a system A, B is stabilizable (controllable or noncontrollable, but the noncontrollable part is stable) and if a system A, C does not have a nonobservable subspace containing an eigenvector corresponding to Re(p) = 0, then the matrix E_{12} will be nonsingular.

b. If A, B cannot be stabilized, then E_{12} is singular.

Proof: We shall prove this theorem by assuming the opposite. Let us assume that A, B is stabilizable and A, C has no nonoberservable subspaces containing an eigenvector corresponding to a purely imaginary eigenvalue, while E_{12} is singular, which is a contradiction [QED].

If E_{12} is singular there exists a vector which satisfies

$$E_{12}\,\mu = 0 \tag{4.1}$$

Such a vector will also satisfy

$$E_{22}\mu = \alpha'; \qquad \alpha \neq 0 \tag{4.2}$$

Since E is nonsingular, it is impossible that $\alpha = 0$ because we then get

$$E \begin{vmatrix} 0 \\ \mu \end{vmatrix} = 0 \tag{4.3}$$

For such a vector α, we have

$$\left[\alpha^{*} ; 0\right] M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = -\alpha'^{*} B R_2^{-1} B' \alpha \tag{4.4}$$

On the other hand,

$$\left[\alpha^{*} ; 0\right] M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \mu'^{*} \left[E_{22}'^{*} ; -E_{12}'^{*}\right] M \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \mu \tag{4.5}$$

Since the columns of E_{ij} appear in conjugate pairs, the only difference between E_{ij} and E_{ij}^{*} is change of order of the columns. We define a matrix F_1 by

$$F_1 = \left[\begin{array}{c|c|cc} 1 & & 0 & 0 \\ \hline & & & \\ \hline 0 & & 0 & 1 \\ 0 & & 1 & 0 \end{array}\right]$$

When $F_{1\,kk} = 1$, the column e_k of E_{ij} is real; when $F_{1\,\underline{1}n} = 1$, the columns $\underline{e}_1$ and e_n are complex conjugates; $F_{1\,rs} = 0$ for all the other elements.

Using the matrix F_1 as defined above, we get

$$\left[\alpha^{*} ; 0\right] M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \mu'^{*} F_1 \left[E_{22}' ; -E_{12}'\right] M \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \mu = \mu'^{*} F_1 \left[E_{22}' ; -E_{12}'\right] E \,\mathcal{J}\, E^{-1} \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \mu$$

$$= \mu'^{*} F_1 \mathcal{J}_{12} \mu = -\mu'^{*} F_1 J(j\omega_{nob}) \mu$$

When A, C has no nonobservable eigenvalue p = jw, $J_{nob}(jw) = 0$, we get

$$[\alpha^* ; 0] \cdot M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = 0$$

from this, together with Eq. (4.4) we have

$$\alpha^{*'} B R_2^{-1} B' \alpha = 0 \tag{4.7}$$

$$B' \alpha = 0$$

For a vector α which satifies Eq. (4.7), we get by substitution

$$M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ -A'\alpha \end{bmatrix} \quad (J_{nob}(j\omega) = 0) \tag{4.8a}$$

On the other hand, similarly to Eq. (4.5),

$$M \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = E \mathcal{J} E^{-1} \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \mu = E \begin{bmatrix} J & -J_{nob}(j\omega) \\ -J_{nc}(j\omega) & -J' \end{bmatrix} E^{-1} \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} \mu = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} (-J' \quad) \mu \tag{4.8b}$$

Writing $\mu_1 = + J' \mu$ and $\alpha_1 = A'\alpha$, we get according to Eqs. (4.8a) and (4.8b)

$$E_{12} \mu_1 = 0 \tag{4.9}$$

$$E_{22} \mu_1 = A'\alpha = \alpha_1$$

This is a form similar to that of Eqs. (4.1) and (4.2), with μ_1 and α_1 replacing μ and α.

Similarly, repeating operations (4.4) through (4.7), we get, in analogy with Eq. (4.7)

$$B' \alpha = 0 \tag{4.10}$$

$$B'A \alpha = 0$$

We repeat operations (4.8) through (4.10) n times, so as to define

$$\mu_n = J'\mu_{n-1} \quad ; \quad \alpha_n = A'\alpha_{n-1}$$

and we get a series of equations

$$\begin{aligned} B'\alpha &= 0 \\ B'A\alpha &= 0 \\ &\cdots \\ B'A^{n-1}\alpha &= 0 \end{aligned} \tag{4.11}$$

Hence α will be nonzero only if α belongs to the space of noncontrollable state variables. We write A and B in canonical form

$$A = \left[\begin{array}{c|c} A_c & \cdots \\ \hline 0 & A_{nc} \end{array}\right], \qquad B = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right]$$

The meaning of Eq. (4.11) is that $\alpha \neq 0$ only if it belongs to the form

$$\alpha = \left[\begin{array}{c} 0 \\ \hline \alpha_{nc} \end{array}\right] \tag{4.12}$$

where α_{nc} corresponds to noncontrollable state variables.

If A is stabilizable, then A_{nc} is unconditionally stable and all the eigenvalues satisfy $\mathrm{Re}(p) \leq 0$.

Since

$$M \begin{vmatrix} E_{12} \\ \hline E_{22} \end{vmatrix} \mu = M \begin{vmatrix} 0 \\ 0 \\ \hline 0 \\ \alpha_{nc} \end{vmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ -A'_{nc}\alpha_{nc} \end{array}\right] \tag{4.13}$$

it follows that

$$e^{Mt} \begin{vmatrix} E_{12} \\ E_{22} \end{vmatrix} \mu = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ e^{-A_{nc}t}\alpha_{nc} \end{array}\right] \tag{4.14}$$

contains only exponents with positive real parts.

As compared to this, $\begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix}$ is, by definition, constructed of eigenvectors corresponding to eigenvalues with only nonpositive real parts. Therefore,

$$e^{Mt}\begin{bmatrix} E_{12} \\ \\ E_{22} \end{bmatrix}\mu$$

contains nonpositive exponents only. Therefore

$$\mu = 0$$
$$\alpha = 0$$

which contradicts Eq. (4.4).

b. We now prove the inverse theorem. Let us assume that E_{12}^{-1} exists, while the system A, B cannot be stabilized. It will be seen that this results in a contradiction.

If A, B cannot be stabilized, then A_{nc} has at least one unstable eigenvalue. This eigenvalue corresponds to an eigenvector z in such a manner that $Re(p) > 0$ and $A'_{nc} z = pz$. We construct the vector

$$\bar{z} = \begin{bmatrix} 0 \\ 0 \\ - \\ 0 \\ z \end{bmatrix} \tag{4.15}$$

which will be an eigenvector of M:

$$M\bar{z} = -p\bar{z} \tag{4.16}$$

Since $Re(p) > 0$, $\bar{z} \in I(-)$. Since the upper part of $\bar{z}$ is equal to zero, E_{12} is nonsingular, which is a contradiction.

From the manner in which this proof was presented, we can deduce a number of corollaries.

Corollary 4.1.1 The columns of E_{12} which do not correspond to nonstabilizable eigenvalues of A and B and which do not correspond to nonobservable eigenvalue p, $Re(p) = 0$ of A, are linearly independent.

Corollary 4.1.2.

$$\mathcal{N}(E_{12}) \supset \qquad (4.17)$$

is a subspace of nonstabilizable state variables

Corollary 4.1.3 If there exists a vector μ,

$$\mu \in \qquad (4.18)$$

is a subspace of nonstabilizable state variables, then 20a is also a subspace of nonstabilizable state variables.

Proof: It can be seen from Eq. (4.15) that the structure of E_{12} is of the form

$$\begin{bmatrix} E_{12c}' ; & 0 \end{bmatrix}$$

where the zero right-hand side of the matrix corresponds to eigenvectors corresponding to nonstabilizable eigenvalues of A. If the vector μ belongs to a subspace of nonstabilizable statevariables, it has the form $\begin{bmatrix} 0 \\ \mu_{nc} \end{bmatrix}$ and satisfies the expression

$$|E_{12}| \mu = 0$$

From Eqs. (4.9) and (4.13)

$$E_{22} \mu = \alpha = \begin{bmatrix} 0 \\ \alpha_{nc} \end{bmatrix}$$

As can be seen, this vector also belongs to the subspace of nonstabilizable state variables.

Existence theorem 4.2

The matrix E_{11} is nonsingular if and only if the noncontrollable part of A (with respect to A and B) is unconditionally nonstable (all its eigenvalues satisfy Re(p) $>$ 0). The proof of this is similar to that of existence theorem 4.1.

a. Let us assume that there exists a vector $\eta \neq 0$ such that $E_{11} \eta = 0$.

Also in this case $E_{21} \eta = \beta$ and $\beta \neq 0$, since E^{-1} exists (similar to Eq. (4.3).

It is known by substitution that

$$[\beta'^* ; 0] \, M \begin{bmatrix} 0 \\ \beta \end{bmatrix} = -\beta^* B R_2^{-1} B' \beta$$

while, on the other hand, it is known that

$$[\beta'^{*};0]M\begin{bmatrix}0\\ \beta\end{bmatrix} = -\eta^{*}F_1\,[-E'^{*}_{21};E'^{*}_{11}]M\begin{bmatrix}E_{11}\\ E_{21}\end{bmatrix}\eta = -\eta^{*}F_1\,J_{nc}(j\omega)\,\eta \geq 0$$

from which it follows that

$$-\beta'^{*}BR_2^{-1}B'\beta = 0$$

$$B'\beta = 0$$

It is also found that $J_{nc}(jw)\,\eta = 0$. On the basis of this and considerations similar to those concerning Eq. (4.8), we get

$$\begin{bmatrix}0\\ A'\beta\end{bmatrix} = M\begin{bmatrix}0\\ \beta\end{bmatrix} = \begin{bmatrix}E_{12}\\ E_{22}\end{bmatrix}J$$

Performing operations similar to those of (4.8)-(4.10), we get

$$\beta'^{*}A_1BR_2^{-1}B'A'\beta = 0$$

and similarly to Eq. (4.11)

$$\begin{aligned}B'A'\beta &= 0\\ &\;\;\vdots\\ B'A'^{n}\beta &= 0\end{aligned}$$

Therefore $\beta = 0$, with the exception of the case when β belong to the noncontrollable space of A, B. Also here

$$M\begin{bmatrix}0\\ 0\\ \hline 0\\ \beta\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \hline 0\\ -A'_{nc}\beta\end{bmatrix}$$

We should make a distinction between the three cases:

1. A_{nc} has eigenvalues only of the kind $Re(p) > 0$. In this case,

$$\underset{t \to \infty}{\text{Norm}} \left\{ e^{Mt} \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ \beta \end{bmatrix} \right\} = \text{Norm} \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ e^{-A'_{nc}t}\beta \end{bmatrix} \to 0$$

while, by definition of $\left\| \begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix} \right\|$

$$\underset{t \to \infty}{\text{Norm}} \left\{ e^{MT} \left| \begin{matrix} E_{11} \\ E_{21} \end{matrix} \right| \right\} \neq 0$$

Therefore $\beta = 0$.

2. A_{nc} has eigenvalues only of the kind $Re(p) = 0$. In this case, the vectors composing E_{11} are eigenvectors of A_{nc} (type-c vectors according to parag. 3.3) and therefore cannot be dependent.

3. A_{nc} has eigenvalues of the kind $Re(p) > 0$ and also $Re(p) = 0$. The considerations used in the two previous cases together yield the conclusion that again in this case $\beta = 0$.

b. The inverse theorem will be proven in the same manner as theorem 4.1. Let us assume that E_{11} is nonsingular and that A_{nc} has a stable eigenvalue. There exists an eigenvector y of A_{nc} such that

$$A_{nc}y = py$$
$$Re(p) < 0$$

The vector

$$\bar{y} = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ y \end{bmatrix}$$

is the eigenvector of M for the eigenvalue $-p$. Therefore, $\bar{y}$ belongs to (I+) and E_{11} has a zero column, which is contradictory.

Corollary 4.2.1 Those columns of E_{11} which do not correspond to noncontrollable and unconditionally stable eigenvalues (Re(p) $>$ 0) are linearly independent.

Corollary 4.2.2

$$\mathcal{N}_b (E_{11})' = \{ \tag{4.19}$$

is a subspace of nonstabilizable state variables.

Corollary 4.2.3

$$y \in \mathcal{N}_b(E_{11}) \Longleftrightarrow E_{21} y \in \mathcal{N}_b(E_{11}) \tag{4.20}$$

The proof is similar to that of 4.1.3

<u>Existence theorem 4.3</u>

a. The matrix E_{21} is nonsingular if the nonobservable part of A (with respect to A, C) is unconditionally instable and there is no eigenvector belonging to the noncontrollable subspace of A and corresponding to an eigenvalue Re(p) = 0.

b. If the nonobservable part of A is unconditionally unstable, then E_{21} will be singular.

Proof: the proof is based on that of existence theorem 4.1. The matrix M' has properties similar to those of M.

The eigenvector matrix of M'' is

$$E\{M'\} = E'^{-1} = \begin{bmatrix} E_{22} & -E_{21} \\ -E_{12} & E_{11} \end{bmatrix}$$

It is clear that everything that has been said with respect to E_{12} and A, B, and C also applies to E_{21} and A', C', B', from which we get theorem 4.3.

Also the corollaries and their proofs will be similar.

Corollary 4.3.1 The columns of E_{21} which do not correspond to nonobservable and stable eigenvalues and which do not correspond to an eigenvalue Re(p) = 0 corresponding to the noncontrollable part of A are linearly independent.

Corollary 4.3.2

$$\mathcal{N}_b(E_{21}) \supset \tag{4.21}$$

is the substance of nonobservable and unstable state variables ($Re(p) > 0$).

Corollary 4.3.3 If

$$y \in \tag{4.22}$$

is a subspace of nonobservable and unstable state variables, then

$$E_{11} y \in$$

is also a subspace of nonobservable and unstable state variables.

Existence theorem 4.4

The matrix E_{22} is nonsingular if and only if the nonobservable part of A relative to A, C contains unconditionally unstable eigenvalues.

Corollary 4.4.1. The columns of E_{22} corresponding to nonobservable and state eigenvalues of A are linearly independent.

Corollary 4.4.2

$$\mathcal{N}o\,(E_{22}) \tag{4.23}$$

is a subspace of nonobservable and stable state variables.

Corollary 4.4.3.

$$y \in \mathcal{N}o(E_{22}) \Leftrightarrow E_{12} y \in \mathcal{N}o(E_{22}) \tag{4.24}$$

The proof of these theorems and corollaries is obtained in a manner analogous to the proof of theorem 3.

Corollary 4.5. When there are no eigenvalues of the kind $Re(p) = 0$

$$\left.\begin{array}{llll} (a) & \mathcal{N}o(E_{12}) = \mathcal{N}o(E'_{12}) & (c) & \mathcal{N}o(E_{11}) = \mathcal{N}o(E'_{11}) \\ (b) & \mathcal{N}o(E_{22}) = \mathcal{N}o(E'_{22}) & (d) & \mathcal{N}o(E_{21}) = \mathcal{N}o(E'_{21}) \end{array}\right| \tag{4.25}$$

Let us prove (a). From Eq. (3.30),

$$E'_{22} E_{12} - E'_{12} E_{22} = 0$$

We select the vector

$$y \in \mathcal{N}o\,(E_{12})$$

for which

$$E'_{12}E_{22}y = 0$$

Since when there are no eigenvectors of the kind Re(p), we have

$$\mathcal{N}(E_{22}) \cap \mathcal{N}(E_{12}) = 0 \qquad (4.26)$$

Since one of them is related to stable eigenvalues, while the other belongs to unstable eigenvalues, it is impossible that

$$y \in \mathcal{N}(E'_{12})$$

and from this

$$E_{22}y \in \mathcal{N}(E'_{12})$$

But from Eqs. (4.21) and (4.22),

$$E_{22}y \in \mathcal{N}(E_{12})$$

Again, since the condition

$$y \in \mathcal{N}(E_{22})$$

is not satisfied, E_{22} corresponds to an one-to-one transformation within $\mathcal{N}(E_{12})$ Therefore,

$$\mathcal{N}(E_{12}) = \mathcal{N}(E'_{12})$$

All the corollaries are proven similarly.

CHAPTER 5

SOLUTION OF THE EULER-LAGRANGE EQUATIONS

In this chapter we shall attempt to obtain explicit expressions for the solutions obtained from the Euler-Lagrange equations. We shall be aided by putting M in canonical form by means of Eq. (3.5) and by the existence theorems of the previous chapter.

5.1. The relationships between A, B, C, and $\mathcal{J}$. From Eq. (3.22),

$$M = E\mathcal{J}E^{-1} \tag{5.1}$$

$$M = E\begin{bmatrix} J & \\ & -J' \end{bmatrix}E^{-1} + E\begin{bmatrix} \cdot & -J_{nob}(j\omega) \\ -J_{nc}(j\omega) & \end{bmatrix}E^{-1}$$

Recalling Eqs. (4.17) and (4.23),

$$J_{nc}(j\omega) \quad \text{columns of} \in \begin{cases} \mathcal{N}(E_{12}) \\ \mathcal{N}(E_{12}') \end{cases} \tag{5.2}$$

$$J_{nob}(j\omega) \quad \text{columns of} \in \begin{cases} \mathcal{N}(E_{22}) \\ \mathcal{N}(E_{22}') \end{cases}$$

We get after substituting E

$$M = \left[\begin{array}{c|c} E_{11}JE_{22}'+E_{12}J'E'_{21} & -E_{11}JE_{12}'-E_{12}J'E_{11}'-E_{11}J_{nob}(j)E_{11}' \\ \hline E_{21}JE_{22}'+E_{22}J'E'_{21}-E_{22}J_{nc}(j)E_{22}' & -E_{21}JE_{12}'-E_{22}J'E_{11}' \end{array}\right] \tag{5.3}$$

We define

$$E_{22} J' E_{22}^{\dagger} = D \tag{5.4}$$

$$E_{22}' E_{21}' = Y$$

$$E_{22}'^{\dagger} E_{22}' E_{22}'^{\dagger} E_{12}' E_{22} E_{22}^{\dagger} = X$$

$$E_{11} E_{22}' = Z$$

According to Eq. (3.28), we have

$$Y = Y' \tag{5.5a}$$

$$E_{22}' E_{12} = E_{12}' E_{22}$$

$$E_{22}'^{\dagger} E_{22}' E_{12} E_{22}^{\dagger} \doteq E_{22}'^{\dagger} E_{12}' E_{22} E_{22}^{\dagger}$$

Multiplying on the right by $E_{22} E_{22}^{+}$ and on the left by $E_{22}'^{+} E_{22}'$, we get

$$E_{22}'^{\dagger} E_{22}' E_{12} E_{22}^{\dagger} E_{22} E_{22}^{\dagger} \doteq E_{22}'^{\dagger} E_{22}' E_{22}'^{\dagger} E_{12}' E_{22} E_{22}^{\dagger} \tag{5.5b}$$

$$X = X'$$

The form in which it has been written exphasizes the fact that X is the submatrix of $E_{12} E_{22}^{+}$ corresponding to (E_{22}).

$$E_{11} E_{22}' \doteq I + E_{12} E_{21}'$$

$$Z = I + E_{12} E_{22}^{\dagger} E_{22} E_{21}' + E_{12} \left(I - E_{22}^{\dagger} E_{22} \right) E_{21}'$$

The columns of $(I - E_{22}^{+} E_{22})$ belong to $\mathcal{N}(E_{22})$; according to Eq. (4.24), the columns of $E_{12}(I - E_{22} E_{22})$, also belong to $\mathcal{N}(E_{22})$ and to $\mathcal{N}(E_{22}')$ as well. Hence,

$$E_{22}' Z = E_{22}' (I + XY) \tag{5.5c}$$

All the vectors X, Y, Z, D are real expressions. Since E_{ij} has complex columns appearing in conjugate pairs, there exists a matrix F_2 such that

$$\mathrm{Im}\ (E_{ij} F_2) = 0$$

and which has the form

$$F_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2j} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2j} \end{array}\right]$$

a. If $F_{2\,kk} = 1$, column e_k of E_{ij} is real.

b. If $F_{2\,\underline{ll}} = F_{2\,ml} = \sqrt{2}/2$, then columns $e_{\underline{l}}$ and e_m are complex conjugates.

c. If $F_{2\,mm} = -F_{2\,\underline{l}m} = \sqrt{2/2j}$, then columns $e_{\underline{l}}$ and e_m are complex conjugates.

d. In all other cases, $F_{2ij} = 0$.

It should be noted that also for F_2^*:

a. $I_m\ (E_{ij}F_2^*) = 0$

b. $F_2F_2'^* = I$

Since the complex columns of all the E_{ij} correspond,

$$Im\ (E_{ij}E'_{kl}) = Im\ (E_{ij}F_2F_2'^*E'_{kl}) = 0$$

Similary,

$$Im\ (E_{22}F_2F_2'^*J'F_2F_2'^*E_{22}^T) = 0$$

$$Im(X) = 0 \ ; \quad Im(Y) = 0 \ ; \quad Im(Z) = 0 \ ; \quad Im(D) = 0$$

We shall now check the properties of matrices X and Y.

a. Equating the submatrices of M as given by Eqs. (2.11) and (5.3), we get

$$-C'R_1C = E_{21}JE'_{22} + E_{22}J'E'_{21} - E_{22}J_{nc}(j\omega)E'_{22} \qquad (5.7)$$

$$= E_{21}E'_{22}E'^{\dagger}_{22}JE'_{22} + E_{22}J'E^{\dagger}_{22}E_{22}E'_{21} - E^*_{22}F_1J_{nc}(j\omega)E'_{22}$$

$$-C'R_1C = YD' + DY - E^*_{22}F_1J_{nc}(j\omega)E'_{22} \qquad (5.8)$$

F_1 is defined according to Eq. (4.6a). Constructing F_1 according to Eq. (4.6a) and $J_{nc}(jw)$ according to Eq. (3.24), we get

$$F_1J_{nc}(j\omega) \geq 0 \qquad (5.9)$$

$$E^*_{22}F_1J_{nc}(j\omega)E'_{22} \geq 0$$

$J_{nc}(jw)$ exists only if A_{nc} has eigenvalues of the kind $p = jw$. These eigenvalues again belong to D'.

Let us divide the space $\mathcal{C}^n$ into two subspaces: $\mathcal{C}_{nc}(jw)$ is the subspace whose base is defined by the generalized eigenvectors of D' corresponding to the eigenvalue $p = jw$ of A_{nc} and $\mathcal{C}^{\perp}_{nc}(jw)$ is the complement of $\mathcal{C}_{nc}(jw)$.

Each vector in the space $\mathcal{C}^n$ will be decomposed into two components

$$x = x_1 + x_2 \begin{cases} x_1 \in \mathcal{C}_{nc}(j\omega) \\ x_2 \in \mathcal{C}^{\perp}_{nc}(j\omega) \end{cases} \tag{5.10}$$

It follows from the definitions of D and x_1 that $E_{22}\, x_1$ is an eigenvector of D'. Therefore $E_{22}x_1 \in \mathcal{N}(J_{nc}(j\omega))$ whence we get after substitution:

$$-x'^*C'R_1Cx = x_1'^*(YD' + DY)x_1 - x_2'^*(E^*_{22}F_1J_{nc}(j\omega)E'_{22})x_2$$

Due to separation of x_1 from x_2, it is clear that

$$x'^*(C'R_1C - E^*_{22}F_1J_{nc}(j\omega)E'_{22})x \geq 0$$

Let us designate the terms in parentheses by Q_1. This is the value of Q which would have been obtained but for eigenvalues of the kind $p = jw$ of A_{nc}.

From this

$$YD' + DY = -Q_1 \leq 0 \tag{5.11}$$

Since D represents a matrix with eigenvalues having only real nonnegative components, it follows from the augmented Routh-Hurwitz criterion that

$$Y \leq 0 \tag{5.12}$$

b. Again, by equating the submatrices of M we get

$$\begin{aligned}-BR_2^{-1}B' &= E_{11}JE'_{12} - E_{12}J'E'_{11} - E_{11}J_{nob}(j\omega)E'_{11} \\ &= E_{11}E'_{22}E'^{\dagger}_{22}JE'_{22}E'^{\dagger}_{22}E_{12} - E_{12}E^{\dagger}_{22}E_{22}J'E^{\dagger}_{22}E_{22}E'_{11} + \\ &+ E_{11}J(I - E'_{22}E'^{\dagger}_{22})E'_{12} - E_{12}(I - E^{\dagger}_{22}E_{22})J'E'_{11} - E^*_{11}F_1J_{nob}(j\omega)E'_{11}\end{aligned}$$

Multiplying by E^{*}_{22} from both sides, we get rid of terms containing $(I - E^{+}_{22}E_{22})$

$$-E'_{22}BR_2B'E^*_{22} = -E'_{22}(ZD'X + XDZ')E^*_{22} - E'_{22}E^*_{11}F_1J_{nob}(j\omega)E'_{11}E^*_{22}$$

and, after constructing $E_{22}Z$ according to Eq. (5.5c), we get

$$-E_{22}'\left\{BR_2^{-1}B' - X(YD' + DY)X\right\}E_{22}^* = -E_{22}'\left\{D'X+XD+E_{11}^*F_1J_{nob}(j\omega)E_{11}'\right\}E_{22}^* \tag{5.13}$$

$$-E_{22}'\left\{BR_2^{-1}B'+XQ_1X-E_{11}F_1J_{nob}(j\omega)E_{11}'\right\}E_{22}^* = E_{22}'(D'X+XD)E_{22}^*$$

Using considerations similar to (5.10) and (5.11), we see that the expression in parentheses is negative-definite.

Since

$$\mathcal{N}(X) > \mathcal{N}(E_{22}) = \mathcal{N}(D)$$

we have

$$D'X + XD \leq 0$$

And according to the extended Routh-Hurwitz criterion, we get according to Eq. (5.11)

$$X \geq 0 \tag{5.14}$$

5.2 Expansion of e^{Mt}

From Eq. (5.1)

$$\Theta(t) = e^{Mt} = E\, e^{\mathcal{J}t}E^{-1} \tag{5.15}$$

Due to the special structure of $\mathcal{J}$, we get

$$e^{\mathcal{J}t} = \begin{pmatrix} e^{Jt} & -e^{Jt}\int_0^t e^{-J\tau}J_{nob}(j\omega)e^{-J'\tau}\,d\tau \\ -e^{-J't}\int_0^t e^{J'\tau}J_{nc}(j\omega)e^{J\tau}\,d\tau & e^{-J't} \end{pmatrix} \tag{5.16}$$

Using Eqs. (5.15) and (5.16) we shall try to find expressions for $\theta_{21}(t)$ and $\theta_{22}(t)$.

Constructing E, we get similarly to Eq. (5.7):

$$\Theta_{21}(t) = E_{21} e^{Jt} E'_{22} - E_{22} e^{-J't} E'_{21} - E'_{22} e^{-J't} \int_0^t e^{J'\tau} J_{nc}(j\omega) e^{J\tau} d\tau E'_{22}$$
$$= E_{21} E'_{22} E'_{22} e^{Jt} E'_{22} - E_{22} e^{-J't} E_{22} E_{22} E'_{21} -$$
$$- E_{22} e^{-J't} E_{22} \int_0^t E_{22} e^{J'\tau} E_{22} E_{22} J_{nc}(j\omega) E'_{22} E'_{22} e^{J\tau} E'_{22} d\tau$$
$$= E_{22} E^{\dagger}_{22} \left\{ Y e^{D't} - e^{-Dt} Y - e^{-Dt} \int_0^t e^{D\tau} E_{22} J_{nc}(j\omega) E'_{22} e^{D'\tau} d\tau \right\} E'^{\dagger}_{22} E'_{22}$$
$$= E_{22} E^{\dagger}_{22} \left\{ e^{-Dt} (e^{Dt} Y e^{D't} - Y - \int_0^t e^{D\tau} E_{22} J_{nc}(j\omega) E'_{22} e^{D'\tau} d\tau) \right\} E'^{\dagger}_{22} E'_{22}$$
$$= E_{22} E^{\dagger}_{22} \left\{ e^{-Dt} (e^{Dt} Y e^{D't} - Y - \int_0^t e^{D\tau} E^{*}_{22} F_1 J_{nc}(j\omega) E'_{22} e^{D'\tau} d\tau) \right\} E'^{\dagger}_{22} E'_{22}$$

Evaluation of the expression in the braces yields:

$$e^{Dt} \left\{ DY + YD' - E^{*}_{22} F_1 J_{nc}(j\omega) E'_{22} \right\} e^{D't} = -e^{D't} Q e^{D't}$$

We define

$$S(t) = \int_0^t -e^{Dt} Q e^{D't} dt \tag{5.17}$$
$$\Theta_{21}(t) = E_{22} E^{\dagger}_{22} e^{-Dt} S(t) \quad E'^{\dagger}_{22} E'_{22}$$

Let us now check $\theta_{22}(t)$ to determine the existence (or nonexistence) of the conjugate point

$$\Theta_{22}(t) = \Theta_{22}(t) E_{22} E^{\dagger}_{22} + \Theta_{22}(t)(I - E_{22} E^{\dagger}_{22})$$

We shall check the first expression only:

$$\Theta_{22}(t) = -E_{21} e^{Jt} E'_{12} + E_{22} e^{-J't} E_{11}$$
$$\Theta_{22}(t) E_{22} E^{\dagger}_{22} = -E_{21} E'_{22} E'_{22} e^{Jt} E'_{22} E'^{\dagger}_{22} E'_{12} E_{22} E^{\dagger}_{22} + E_{22} e^{-J't} E^{\dagger}_{22} E_{22} E'_{11} E_{22} E^{\dagger}_{22}$$
$$\Theta_{22}(t) E_{22} E^{\dagger}_{22} = \left\{ -Y e^{D't} X + e^{-Dt} Z' \right\} E_{22} E^{\dagger}_{22}$$
$$= e^{-Dt} \left\{ -e^{Dt} Y e^{D't} X + I + YX \right\} E_{22} E^{\dagger}_{22}$$
$$= e^{-Dt} \left\{ I - (e^{Dt} Y e^{D't} - Y) X \right\} E_{22} E^{\dagger}_{22}$$

Differentiating the expression in the parentheses, we get

$$\frac{d}{dt}\left(e^{Dt}Ye^{D't}-Y\right) = e^{Dt}\left\{DY+YD'\right\}e^{D't} = -e^{Dt}Q_1e^{D't}$$

Q_1 is defined by Eq. (5.1). We now define S_1 in terms of Q_1

$$S_1 = \int_0^t e^{Dt}Q_1e^{D't}dt \tag{5.18}$$

and get

$$\Theta_{22}(t)E_{22}E_{22}^{\dagger} = e^{Dt}\left\{\left[I + S_1 X\right]E_{22}E_{22}^{\dagger}\right\}$$

The second expression for $\theta_{22}(t)$ is

$$\begin{aligned}\Theta_{22}(t)\left\{I-E_{22}E_{22}^{\dagger}\right\} &= (-E_{22}e^{Jt}E'_{12}+E_{22}e^{-J't}E_{11})(I-E_{22}E_{22}^{\dagger})\\ &=(-E'^{\dagger}_{22}E'_{22}E_{21}e^{Jt}E'_{12}+E_{22}e^{-J't}E_{11})(I-E_{22}E_{22}^{\dagger})\\ &\quad-(I-E'^{\dagger}_{22}E'_{22})E_{21}e^{Jt}E'_{12}(I-E_{22}E_{22}^{\dagger})\end{aligned}$$

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$$\Theta_{22}(t)=\underbrace{e^{D'}\left\{I+S_1X\right\}}_{\Theta_{22}(t)_{11}}+\underbrace{-E'^{\dagger}_{22}E'_{22}E_{21}e^{Jt}E'_{12}+E_{22}e^{-J't}E'_{11}}_{\Theta_{22}(t)_{12}}(I-E_{22}E_{22}^{\dagger})$$

$$+\underbrace{(I-E_{22}E_{22})E_{21}e^{Jt}E'_{12}(I-E_{22}E_{22})}_{\Theta_{22}(t)_{22}}$$

$\theta_{22}(t)$ has three parts, for which it can be stated that

$$\begin{aligned}&\mathcal{N}(\Theta_{22}(t)_{11})=\mathcal{N}(E_{22}) && \mathcal{R}(\Theta_{22}(t)_{11})=\mathcal{R}(E_{22})\\ &\mathcal{N}(\Theta_{22}(t)_{12})=\mathcal{R}(E_{22}) && \mathcal{R}(\Theta_{22}(t)_{12})=\mathcal{R}(E_{22})\\ &\mathcal{N}(\Theta_{22}(t)_{22})=\mathcal{R}(E_{22}) && \mathcal{R}(\Theta_{22}(t)_{22})=\mathcal{N}(E_{22})\end{aligned}$$

The structure of the matrices $\theta_{22}(t)_{ij}$ is such that each of them has three zero submatrices and one nonzero submatrix $\theta'_{22}(t)_{ij}$. Accordingly, the structure of the matrix $\theta_{22}(t)$ is

$$\Theta_{22}(t) = \begin{bmatrix} \Theta'_{22}(t)_{11} & \Theta'_{22}(t)_{12} \\ 0 & \Theta'_{22}(t)_{22} \end{bmatrix} \tag{5.19}$$

The columns of $(I - E_{22}E_{22}^{+})$ as well as those of $E'_{12}\ (I - E_{22}E_{22}^{+})$ belong to $\mathcal{N}(E_{22})$ and hence cannot belong to $\mathcal{N}(E_{21})$. Therefore, for any vector that satisfies $y \in \mathcal{N}(E_{22})$, it is also true that $\theta'_{22}\ (t)_{22} y \neq 0$, whence

$$\det \Theta'_{22}(t)_{22} \neq 0$$

Since $\mathcal{R}(\Theta_{2211}) = \mathcal{R}(E_{22})$ while according to Eq. (5.19) the structure of $\theta_{22}^{-1}(t)$ is

$$\Theta_{22}^{-1}(t) = \begin{bmatrix} \Theta_{22}^{-1}(t)_{11} & ---- \\ 0 & \Theta_{22}^{-1}(t)_{22} \end{bmatrix}$$

We get

$$P(t) = E_{22}E_{22}^{\dagger}\left\{I+S_1(t')\Phi\right\} e^{-Dt}e^{Dt}S(t)E_{22}E_{22}^{\dagger}$$

$$P(t) = E_{22}E_{22}^{\dagger}\left[I+S_1(t)X\right]^{\dagger} S(t)$$

Since the difference between S(t) and S_1 (t) consists in the expression $E_{22}\ J_{nc}(jw)$ E'_{22} and

$$\mathcal{N}(X) \supset \mathcal{N}(E_{12}) \supset$$

is a subspace of nonstabilizable state variables, we have

$$S_1X = SX$$

and

$$P(t) = E_{22}^{\dagger}E_{22}\left\{I+S(t)X\right\} S(t)E'_{22}E'^{\dagger}_{22} \tag{5.20}$$

5.3 Solution as $t_f \to \infty$

Let us check (S). By definition of D and S, it is clear that

$$\mathcal{N}(S) \supset$$

is a subspace of nonobservable and stable variables.

Also, if A_{nob} has an eigenvalue, then the eigenvector corresponding to it will also be the eigenvector of D', and therefore $\mathcal{N}(S)$ will also contain a subspace of nonobservable and unstable state variables. Therefore,

$$\mathcal{N}(S) \supset$$

is a subset of nonobservable state variables.

$S^{+}(t) \to 0$ as $t \to \infty$ since D' corresponds to an unstable matrix.

Let us find $\lim_{t\to\infty} p(t)$

$$\begin{aligned}
\lim_{t\to\infty} P(t) &= E_{22}E_{22}^{+}\left\{I+\lim_{t\to\infty}S(t)X\left\{\lim_{t\to\infty}S^{+}(t)\right\}^{+}\right\}^{+} \\
&= E_{22}E_{22}^{+}\left\{\lim_{t\to\infty}S^{+}(t)+S^{+}(t)S(t)X\right\}^{+}\lim_{t\to\infty}S^{+}(t)S(t)E_{22}E_{22} \\
&= \left\{S^{+}(t)S(t)X\right\}^{+}S^{+}(t)S(t)
\end{aligned}$$

$\lim_{t\to\infty} P(t)$ will have a finite value, except when

$$\mathcal{N}(X) \not\subset \mathcal{N}(S)$$

On the other hand,

$$\mathcal{N}(X) \supset \left\{\right.$$

is a subspace of nonstabilizable state variables.

P(t) will have no solution when A has a nonstabilizable and nonobservable eigenvalue. In all the other cases, P(t) is finite and equal to

$$P(t) = S^{+}(t)S(t)X^{+}S(t)S^{+}(t)$$

CHAPTER 6

THE PROBLEM OF THE MAXIMUM

6.1 Introduction

The advantage of the method used here for solving the problem of the minimum is that it can be adapted to the solution of similar problems. We shall use this method for solving the problem of the maximum for $R_1 < 0$. In order to avoid repetition, we shall not again present the proof but shall refer to the applicable proofs in the problem of the minimum while pointing out the differences that exist.

6.2 The Eigenvalues of M

The symmetry properties of the eigenvalues are independent of the sign of R_1; hence, theorems 5-7, which were proven in Chapter 3, apply also here.

Unlike the problem of the minimum, in the problem of the maximum it is possible for M to have eigenvalues of the kind Re(p) = 0 even if the system A does not have eigenvalues of this kind. Let us therefore differentiate between two kinds of pure imaginary eigenvalues.

a. Eigenvalues with zero real part belonging to the noncontrollable or non-observable part of A (eigenvalues of the closed loop). These values will always appear with an even multiplicity.

b. Eigenvalues with zero real part which are obtained from the optimum solution (eigenvalues of the closed loop).

Theorem 8 also applies to type-a purely imaginary eigenvalues whereas for the others we get the following theorem.

Theorem 8a The matrix M will not have additional eigenvalues of the kind Re(p) = 0 (of the closed loop) if and only if

$$x^{*\prime}\left\{(A-j\omega)(C'R_1C)^t\ (A'+j\omega)+BR_2^{-1}B'\right\}x<0 \tag{6.1}$$

$$x:\ (A'+j\omega)x\in\mathcal{R}\,(C'R_1C)$$

$$\forall\ \omega$$

The proof of this theorem follows from Eq. (3.12).

To the extent that there exist zero eigenvalues (of type b) which are the solution of Eq. (3.13), they will satisfy the following theorem:

<u>Theorem 9</u> If M has eigenvalues $p = 0$ which do not correspond to a type-b eigenvalue of A, then for each ordinary eigenvector corresponding to this eigenvalue there will exist a general eigenvector.

Proof: From Eq. (6.1), the upper part of the simple eigenvector corresponding to $p = 0$ satisfies the equation

$$x'^{*}\left\{A(C'R_1C)^{+}A'+BR_2^{-1}B'\right\}x = 0 \tag{6.2}$$

Where the lower part of the vector is obtained uniquely from its upper part (Eq. (3.10).

The evenness of multiplicity of the eigenvalues of M ensures that, if there exist m vectors satisfying Eq. (6.2), then the multiplicity of eigenvalue $p = 0$ will be 2m. Therefore, there must exist additional nonordinary (generalized) eigenvectors.

We shall put M in canonical form here in the same manner as this was done in Eq. (3.5) except that while in the case of the problem of the minimum the imaginary eigenvalues appeared in conjugate pairs, the additional imaginary eigenvalues (of type b), when these exist, may also appear in single pairs. In such a case, we shall arbitrarily assign the eigenvalues with zero real part and positive imaginary part to group $I_{(+)}$, while the others will belong to group $I_{(-)}$. Should there exist a zero-type eigenvalue, we shall assign the ordinary eigenvectors to group $I_{(-)}$ and the generalized eigenvectors to group $I_{(+)}$.

We get

$$E^{-1}ME = \mathcal{J} \tag{6.3}$$

$$E = TE'^{-1}T^{-1} \tag{6.4}$$

where

$$\mathcal{J} = \begin{bmatrix} J & -J_{nob}(j\omega) \\ J_{nc}(j\omega)-J(0) & -J' \end{bmatrix} \tag{6.5}$$

with $J(0)$ defined similarly to $J_{nc}(jw)$ for a noncontrollable eigenvalue.

6.3 Existence theorems

The existence theorems which were proven in Chapter 4 apply also here with minor modifications. We now consider the case that, with the exception of $p = 0$, M has no additional eigenvalues $Re(p) = 0$ of an additional kind (type-b). The proof of this can be extended to other cases but will not be given.

Existence theorem 4.1 applies in its previous form;

Existence theorem 4.2 applies in its previous form;

Existence theorems 4.3 and 4.4 acquire the following forms:

Existence theorem 4.3a. The matrix E_{21}' is nonsingular if and only if the nonobservable part of A relative to A, C is unstable.

Existence theorem 4.4a a. Matrix E_{22} is nonsingular if the nonobservable part of A relative to A, C contains unconditionally unstable eigenvalues ($Re(p) > 0$), if there does not exist an eigenvector belonging to the noncontrollable subspace of A and corresponding to an eigenvalue of the kind ($Re(p) = 0$, and if there does not exist a type-b eigenvalue $p = 0$.

b. If the nonobservable part of A contains stable eigenvalues, then E_{22} is singular.

6.4 Solution of the Euler-Lagrange Equations

The solution of these equations for the case of a maximum is the same as the problem of a minimum, and the development is therefore also the same. It should only be remembered that it is possible here to have complex X, Y, Z, and D.

For convenience in proof, we shall assume that the system is stabilizable and observable. This assumption does not detract from the generality of the approach, since it is clear that in the case when the system is not stabilizable, the solution as $t_f \to \infty$ will be infinite, while for the nonobservable part it was shown before that its properies have no bearing on the existence or nonexistence of a solution. We make a distinction between two cases.

6.4.1 M has no type-b eigenvalues with zero real part. Similarly to Eq. (5.19) we get

$$\Theta_{22}(t) = e^{-Dt} [I + S(t) X(t)] \tag{6.7}$$

S(t) is defined as in (5.17) except that now $Q < 0$. Since it was assumed that the system is stabilizable and observable

$$X \neq 0$$

$$\Theta_{22}(t) = e^{Dt} [X^{-1} + S(t)] X$$

S(t) ranges from zero to negative infinity. In order that there be no conjugate point for all t, it is required that

$$[\,X^{-1} + \dot{S}(t)\,] < 0 \tag{6.8}$$

$$X^{-1} < 0 \tag{6.9}$$

We shall now check when an expression such as this is obtained. Expressing the submatrices of A according to Eq. (5.3)

$$A = Z\,D' + X\,D\,Y$$

and making the substitution $Z = I + XY$, we get

$$\begin{aligned} A = D' + X\,(\,DY + YD'\,) &= D' + X\,C'R_1C \\ D' &= A - X\,C'R_1C \end{aligned} \tag{6.10}$$

Substitution of D' into Eq. (5.13) yields

$$-(\,X A + A'X\,) = X\,C'R_1CX - BR_2^{-1}\,B' \tag{6.11}$$

Since $R_1 < 0$, it follows, according to the augmented Routh-Hurwitz criterion, that X will be negative definite only if A is unconditionally stable.

Hence the necessary condition for existence of a solution to the problem of the maximum for any t is that A be stable.

The solution of the Riccati equation is

$$P = X^{-1} \tag{6.12}$$

It can be seen from Eq. (6.11) that this expression solves the Riccati equation. It should be noted that the matrix $\overline{X} = E_{11}E_{22}^{+}$ is symmetrical and then $\overline{X}^{-1}$ will be a solution of the Riccati equation and yield a different solution than X^{-1}; i.e., in this case the Riccati equation has two solutions.

It can be seen from Eq. (6.10) that

$$A - X\,C'RC = D' \tag{6.13}$$

Hence Eq. (6.12) corresponds to an unconditionally unstable matrix.

It can be shown that $A - \overline{X}(C'R_1C)$ corresponds to an unconditionally stable matrix. Hence, of the two solutions which will be found for the Riccati equation, we select the one that satisfies

$$\begin{aligned} &\text{Eigenvalues of } [\,A - X\,(C'R_1C)\,] \quad 0 \\ &\quad\text{"} \quad\quad \text{"} \quad\quad \text{"} \; [\,A - P^{-1}\,(C'R_1C)\,] \quad 0 \end{aligned} \tag{6.14}$$

When A is unstable, the maximum t for which it is possible to solve the problem of the maximum is the smallest t for which

$$\det [\, X^{-1} + S(t) \,] = 0 \qquad (6.15)$$

6.4.2. M has kind b purely imaginary eigenvalues

As in the preceding section, we shall assume that the system A is stabilizable and observable. Due to the cumbersome expressions which are obtained, it is difficult to obtain explicit expressions, and hence we shall attempt to examine this case by comparison with the preceding case. We prove the following theorem:

Theorem 10 We are given a system defined by matrices A, B, and C and the cost functions

$$f(R_1) = \int_0^{t_f} (y' R_1 y + u' R_2 u)\, dt$$

A necessary and sufficient condition for a finite solution of $f(R_1)$ is that the problem have a solution for the case of cost function $f(mR_1)$

$$m \lesssim 1$$

Proof: Let us assume that $f(mR_1)$ has no finite solution, i.e., that is, it always possible to find a u(t) such that $f(mR_1) \to \infty$. For such u(t), we also have $f(R_1) \to \infty$, since

$$|f(mR_1); t_f)| \lesssim |f(R_1; t_f)|$$

Therefore there will be no finite solution for $F(R_1, t_f)$. The proof of the inverse of this theorem is identical.

We now employ this theorem for drawing conclusions regarding the case at hand. In our case, the equation

$$x^{*\prime} [\, (A - j\omega I)(C' R_1 C)^{\dagger} (A' + j\omega I) + B R_2^{-1} B'] \, x = 0$$

has a solution for x. It is always possible to replace R_1 by mR_1, when m is such that

$$x^{*\prime} [\, (A - j\omega I) m^{-1} (C' R_1 C)^{\dagger} (A' + j\omega I) + B R_2^{-1} B'] \, x = 0$$

has solutions of the kind Re(p) = 0 only for x = 0. It was seen before that in such a case we get an infinite solution as $t_f \to \infty$. Hence, according to theorem 10, it will be infinite in our case also.

6.5 Results

We have seen that, unlike the problem of the minimum, the problem of the maximum can be solved for all the t_f only in particular cases. The criteria for existence of a solution are quantitative rather than qualitative.

The conditions obtained as $t_f \to \infty$ are more of the qualitative kind.

The sufficient and necessary condition for the existence of a solution to the problem of the maximum as $t_f \to \infty$ is: a) M has no complex eigenvalues; b) the system A is unconditionally stable.

Condition a) can be checked out according to theorem 8. From this theorem we get the necessary condition for the existence of a solution

$$A Q A' - B R_2^{-1} B' \quad 0$$

It should be noted that this condition is sufficient when AQ is symmetrical. When Q = −I, this shows that it is easier to obtain a finite solution when A is symmetrical and, conversely, it is easier to obtain an infinite solution when A has an increasing asymmetry.

<u>Examples for the Solution of the Problem of the Maximum</u>

We not present three numerical examples of the solution of the problem of the maximum ($R_1 < 0$). These examples pertain to first-order systems and are suitable for three cases characteristic of the problem of maximum.

Example 1. We are given the system

$$\dot{x} = 5x + u$$

and the cost function

$$f(x;u) = \int_0^{t_f} (-9x^2 + u^2)\, dt$$

which is to be maximized [sic].

For this problem,

$$A = 5 \quad B = 1 \quad C = 1$$
$$R_1 = -9 \quad R_2 = 1$$

The matrix of Euler-Lagrange equations for this case will be

$$M = \begin{vmatrix} 5 & -1 \\ 9 & -5 \end{vmatrix}$$

$$\det\ [\ M - sI\ ] = (\ 5 - s\ )\ (\ -5 - s\ ) = s^2 - 16$$

The eigenvalues of M are $s_1 = +4$; $s_2 = -4$. Diagonalization of M yields

$$M = \begin{bmatrix} 5 & -1 \\ 9 & -5 \end{bmatrix} = \begin{bmatrix} 9/\sqrt{8} & -1/\sqrt{8} \\ -1/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}^{-1} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 9/\sqrt{8} & -1/\sqrt{8} \\ -1/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}$$

$$\Theta(t) = e^{Mt} = \begin{bmatrix} 9/\sqrt{8} & -1/\sqrt{8} \\ -1/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}^{-1} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 9/\sqrt{8} & -1/\sqrt{8} \\ -1/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}$$

$$\Theta(t) = \begin{bmatrix} 1/\sqrt{8} & 1/\sqrt{8} \\ 1/\sqrt{8} & 9/\sqrt{8} \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 9/\sqrt{8} & -1/\sqrt{8} \\ -1/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}.$$

To check for existence of a conjugate point we must check out the matrix

$$\Theta_{22}(t) = \frac{1}{8}\ [-e^{4t} + 9\ e^{-4t}\ ]$$

This expression vanishes for $t_c = \frac{1}{8} \ln 9 \approx 0.27$, and hence there exists a conjugate point for the problem of the maximum for $t_f > t_c$. An optimum solution will exist only for $t_f < t_c$, when there is no conjugate point.

Since

$$\Theta_{21}(t) = \frac{9}{8}\ (\ e^{4t} - e^{-4t})$$

it follows that, in cases when a solution does exist, the cost function will have the value, according to Eq. (2.19),

$$P\ (\ x_o\ ) = \frac{e^{4t_f} - e^{-4\ t_f}}{-e^{4t_f} + 9e^{-4\ t_f}}\ x_o^{\ 2}$$

Let us now try to gain insight into the meaning of the conjugate point. Let us assume that the system has the initial condition $x_0 = 0$, if an optimum solution exists, when we shall get for the cost function $P(0) = 0$. Had we selected $u = u_0$ = const, we would have obtained

$$\dot{x} = 5x + u_o$$

$$x = \frac{1}{5} \left(e^{5t} - 1 \right) u_o$$

$$f(x;u) = \int_o^{t_f} (-9x^2 + u^2)\, dt = \int_o^{t_f} \left[-\frac{9u_o^2}{25} \left(1-e^{5t} \right)^2 + u_o^2 \right] dt$$

$$= u_o^2 \left[\frac{16}{25} t_f + \frac{18}{125} e^{5t_f} - \frac{9}{250} e^{10t_f} \right]$$

The expression in the brackets is positive for $t < 0.32$ and negative for $t > 0.32$, and when $u_0 \to \infty$ this expression approaches $-\infty$. That is, it is possible to obtain a better minimization for $t > 0.32$ than that which would be obtained with optimum u. It should be noted that in fact the value $t = 0.32$ is higher than that of the conjugate point (which is $t_c = 0.27$); this means that the selection of u_0 = const for $0.27 < t_f < 0.32$ is poor. However, it is possible to find another u(t) which will make the cost function go to infinity.

Examination of the cost function shows that the optimum cost function changes sign at t_c and becomes positive for $t_f > t_c$. Selection of $u = 0$ will always result in a minimal value.

Example 2. We are given the system

$$\dot{x} = -5x + u$$

with a cost function identical to that of example 1

$$f(x;u) = \int_o^{t_f} (-9x^2 + u^2)\, dt$$

Here

$$A = -5 \qquad B = 1 \qquad C = 1$$

$$R = -9 \qquad R_f = 1$$

The matrix of the Euler-Lagrange equation is

$$M = \begin{bmatrix} -5 & -1 \\ 9 & +5 \end{bmatrix}$$

$$\det (M - sI) = s^2 - 16$$

Again in this case the eigenvalues are $s_1 = +4 \qquad s_2 = -4$

Again, by diagonalizing we get for the system's matrix

$$\Theta(t) = e^{Mt} = \begin{bmatrix} -1/\sqrt{8} & -1/\sqrt{8} \\ 9/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}^{-1} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} -1/\sqrt{8} & -1/\sqrt{8} \\ 9/\sqrt{8} & 1/\sqrt{8} \end{bmatrix}$$

For $\theta_{22}(t)$ we get

$$\Theta_{22}(t) = -\frac{1}{8}\left[-e^{-4t} + 9e^{4t}\right]$$

This expression is never zero, and hence there will be no conjugate point for any finite t. Hence,

$$\Theta_{21}(t) = 9e^{4t} - 9e^{-4t}$$

And according to Eq. (2.19)

$$f(x_o) = P(t)\, x_o^2 = -\frac{9e^{4t_f} - 9e^{-4t_f}}{-e^{-4t} + 9e^{4t_f}}\, x_o^2$$

As $t_f \to \infty$,

$$P(t_f)\Big|_{t_f \to \infty} = -1$$

It is expected that there exists a solution for t_f, since A is stable, while M has no purely imaginary eigenvalues. It should be noted that $P = -1$ is a solution of the applicable Riccati equation

$$9 + 10\,P + P^2 = 0$$

Also $P = -9$ is a solution in this case. This solution, however, is incorrect. To show that this is so, we apply the test established in Eq. (6.12)

The eigenvalues of $[A - P^{-1}(C'R_1C)]$ must be positive for

$$-5 - (+1)(-9) = 4 \qquad\qquad P = -1$$

$$-5 - (+\tfrac{1}{9})(-9) = -4 \qquad\qquad P = -9$$

and hence only $P = -1$ is the correct solution.

Example 3. We are given the system

$$\dot{x} = 3 \cdot x + u$$

and an eigenfunction for minimization

$$f\ (x\ ;\ u) = \int_0^{t_f} (-25\ x^2 + u^2)\ dt$$

In this case,

$$A = 3 \quad B = 1 \quad C = 1 \quad R_1 = -25 \quad R_2 = +1$$

The matrix M which is obtained is

$$M = \begin{bmatrix} 3 & -1 \\ 25 & -3 \end{bmatrix};\ \det\ (M - sI) = s^2 + 16$$

and the eigenvalues are $s = \pm j4$. Expansion of e^{Mt} yields

$$(t) = e^{Mt} = \begin{bmatrix} 1 & 1 \\ 3-j4 & 3+j4 \end{bmatrix} \begin{bmatrix} e^{j4t} & 0 \\ 0 & e^{-j4t} \end{bmatrix} \begin{bmatrix} 3+j4 & -1 \\ -(3-j4) & 1 \end{bmatrix}$$

After expansion, we get for $|O'_n(t)$

$$_{22}(t) = \frac{3}{4}\ (\sin 4t + \cos 4t) = \frac{5}{4}\ [\sin(4t + \phi\ )]$$

$$\phi = \arc \sin \frac{4}{5}$$

which becomes zero for the first time when

$$4t + \phi = 2\pi$$

$$T_c = \frac{1}{4}\ (\ 2\pi - \arc \sin \frac{4}{5}\ )$$

and this is the maximum value that t_f may attain.

It can be seen that in the case of time variations the solution is always smaller than one cycle.

From the cover letter

..... The existence theorems in Chapter 4 can also be proven as inverse theorems in all the cases. I did not want to complicate matters and hence did not include these proofs. The proofs are by means of substitution.

The development in Chapter 5 is cumbersome and nonelegant, but this was done for the sake of mathematical exactness. The extensive use of $E_{22}\ E_{22}^{+}$ was caused by the desire to obtain matrices corresponding to the nullity space of E_2. For example, if

$$M = \begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix} \qquad E_{22} E_{22}^{+} = \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix}$$

then

$$E_{22}^{\prime\, I} E_{22}^{\prime\, +} M \; E_{22}' E_{22} = \begin{vmatrix} M_1 & 0 \\ 0 & 0 \end{vmatrix}$$

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